Generalized Offsetting Using a Variable-Radius Voronoi Diagram

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Abstract

We investigate ways to extend offsetting based on skeletal structures beyond the well-known constant-radius and mitered offsets supported by Voronoi diagrams and straight skeletons for which the orthogonal distance of offset elements to their input elements is uniform. We introduce a new geometric structure called the *variable-radius Voronoi diagram*, which supports the computation of variable-radius offsets, i.e., offsets whose distance to the input may vary along the input.

1 Introduction

Offsetting is an important task in diverse applications in the manufacturing business. For a set C in the Euclidean plane, the constant-radius offset with offset distance r is the set of all points of the plane whose minimum distance from C is exactly r. Formally, this offset curve can be defined as the boundary of the set $\bigcup_{p\in C} B(p,r)$, where $B(p,r)$ denotes a disk with radius r centered at the point p.

For polygons such an offset curve will consist of one or more closed curves made up of line segments and circular arcs. Held [2] describes as algorithm using the Voronoi diagram to compute such an offset efficiently and reliably. Mitered offsets differ from constant-radius offsets in the handling of non-convex vertices of an input polygon: Instead of adding circular arcs to the offset curve, the offset segments of the two edges incident to a non-convex vertex get extended until they intersect. This type of offset can be generated in linear time from the straight skeleton [3]; see Figure 1. A common feature of these offsets is that the orthogonal distance of each offset element from its defining contour element is constant.

Several applications in industry, such as for garment manufacture or shoe design, need to construct differently sized pieces from a single master design. The obvious method of scaling is not always desirable as it scales all elements equally. An alternative is to use offsetting, and a common practical requirement is creating non-constant offsets, i.e., offset curves where the distance of each point of the offset to the original input curve changes along the input.

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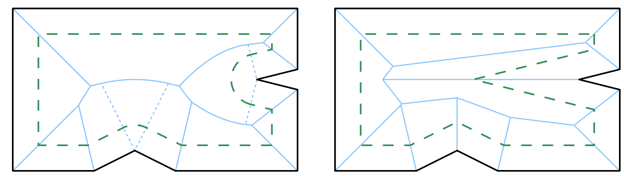

Figure 1: The Voronoi-diagram (left) and the straight skeleton (right) of a polygon enable efficient computation of constant-radius and mitered offsets (dashed).

Prior work on variable-distance offsets [4, 5, 6] seems to concentrate on defining and comparing different offsets and is less concerned with robustly computing offset curves.

2 Main Idea

Consider a planar straight-line graph S in the plane. Let us denote by $\overline{S} \subset \mathbb{R}^2$ the set of points covered by all vertices and line segments of S. Furthermore, we consider a weight function $\sigma\colon \overline{S} \to \mathbb{R}_+$ that assigns to each vertex p of S a positive weight $\sigma(p)$ and for each point on a line segment $\overline{pq}$ of S we linearly interpolate its weight along $\overline{pq}$ from $\sigma(p)$ at p to $\sigma(q)$ at q.

We now place a disk at each point p of $\overline{S}$. In analogy to the so-called prairie fire model, all disks have initially radius zero. As time increases, however, the radius of each disk grows proportional to the weight $\sigma(p)$ of its center point $p \in \overline{S}$. The *variable-radius offset* for a given time is the envelope of this set of disks. As intended, input sites with small weight will induce an offset that is closer to them, and input sites that were assigned larger weights will cause their offsets to be farther away. Formally, this offset is the boundary of the set $\bigcup_{p\in\overline{S}} B(p,\sigma(p)\cdot t)$. Note that the term $\sigma(p)\cdot t$ replaces the constant radius r of standard offsets.

Having used skeletal structures such as the Voronoi diagram and the straight skeleton to construct constant-radius and mitered offsets in the past, we are looking for another Voronoi-like structure to facilitate the computation of non-constant offsets. The variable-radius Voronoi diagram introduced below and defined relative to weighted points and variably-weighted line segments is such a useful structure.

Preliminaries. The Voronoi diagram $\mathcal{VD}(S)$ of a set S of points in the plane, called *sites*, tessellates the plane into interior-disjoint *regions*. Each Voronoi region belongs to exactly one site s and is the locus of all points in the plane whose closest site is s.

We introduce the *variable-radius Voronoi diagram* $\mathcal{VD}_v(S)$ as a generalized Voronoi diagram with generalizations into two directions: First, the set S of input sites is a planar straight-line graph, i.e., a set of both vertices and non-intersecting line segments between pairs of these vertices. Second, we assign multiplicative weights to these sites. As described above, vertices $s \in S$ are assigned positive weight $\sigma(s)$, and the weight of a point on a line segment $\overline{pq}$ changes linearly between its endpoints from $\sigma(p)$ to $\sigma(q)$.

The distance of a point u in the plane to a vertex site s is defined as the Euclidean distance from u to s, divided by the weight of that site: $d(u, s) := \frac{\|u-s\|}{\sigma(s)}$. The distance of u to a line-segment site $\overline{pq}$ is naturally defined as the minimum distance of u to any point of the line segment: $d(u, \overline{pq}) := \min_{v\in\overline{pq}} \frac{\|u-v\|}{\sigma(v)}$.

As in the case of the standard Voronoi diagram, every point in the plane is in the (generalized) Voronoi region of the site that it is closest to. An arc that separates two regions comprises all points that have the same distance to two sites and a larger distance to all other sites.

The variable-radius Voronoi diagram inherits several important properties from the multiplicatively weighted Voronoi diagram of points. For instance, the region of a given site need not be connected and bisectors between two vertices are circles or circular arcs [1]. Other bisectors, however, are more complex curves in general.

The bisectors between a line segment $\overline{pq}$ and its two incident vertices p and q exhibit an interesting property: We can show that they are full circles whose diameters on the line supporting $\overline{pq}$ are bounded by a common point on one side and p and q, respectively, on the other; see Figure 2, left.

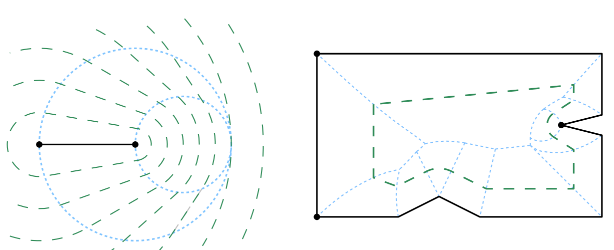

Figure 2: (Left) The variable-radius Voronoi diagram (blue, dotted) of a line segment and its two incident vertices. A family of offset curves is shown in green and dashed. (Right) The variable-radius Voronoi diagram inside a polygonal input with weighted vertices.

Offsetting. While the bisectors of $\mathcal{VD}_v(S)$ consist also of non-trivial curves, it can be shown that the variable-radius offset itself comprises line segments and circular arcs only; see Figure 2, right.

We can compute the variable-radius offset of S for a given time t from the variable-radius Voronoi diagram $\mathcal{VD}_v(S)$. The approach is identical to how constant-distance offsets are computed based on Voronoi diagrams or straight skeletons [2, 3]. Roughly, we iterate through all the arcs of $\mathcal{VD}_v(S)$ and add offset elements in each face that contains points at distance $t \cdot \sigma$. The topological information encoded in $\mathcal{VD}_v(S)$ enables us to do this in time linear in the size of the Voronoi diagram and in a single iteration, without the need to compute all pair-wise self-intersections of offsets.

Construction. Similarly to the standard Voronoi diagram, the variable-radius Voronoi diagram can be obtained from the lower envelope of surfaces in 3D. For vertices, the corresponding surface in 3D is a cone whose dihedral angle depends on the weight of the input vertex. Input line segment induce ruled surfaces in 3D as the offsets of line segments are also line segments. In particular, the surfaces will be subsets of right conoids.

CGAL's 3D envelope computation algorithm is generic in the sense that it can deal with arbitrary terrain surfaces so long as it has some means to learn about certain geometric properties. We are working on proof-of-concept code, based on CGAL, to compute the variable-radius Voronoi diagram of planar straight-line graphs and to compute variable-radius offsets.

3 Conclusion

We investigate one specific variant of a skeletal structure which we call the variable-radius Voronoi diagram. While this structure is of particular interest in itself, we demonstrate its applicability to robustly constructing variable-radius offsets.

An open problem is to generalize the class of input sites to include for instance circular arcs. We hope that this would enable offsets that are $\mathcal{G}^1$ continuous for some class of inputs. However, note that the offset of a variable-weighted circular arc is not a circular arc. Hence, a better understanding of the mathematical characteristics of the resulting offsets and of the corresponding Voronoi bisectors is required.

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